g-OLOGY

Brian Hayes

Science is often portrayed as a kind of tennis match between theory and experiment. This description is nowhere more apt than in the study of a physical constant known as the g factor of the electron and the muon. For more than 50 years g has been batted back and forth by theorists and experimenters striving always to append another decimal place to the known value. The game depends on having well-matched players on either side of the net, so that what's predicted theoretically can be checked experimentally. In this case the players are very good indeed. The g factor of the electron has been both calculated and measured so finely that the uncertainty is only a few parts per trillion. The current experimental value is 2.0023193043718 ± 0.0000000000075.

Measuring a property of matter with such extraordinary precision is a labor of years; a single experiment could well occupy the better part of a scientific career. It's not always appreciated that theoretical calculations at this level of accuracy are also arduous and career-consuming. Getting to the next decimal place is not back-of-the-envelope work. It calls for care and patience and for mastery of specialized mathematical methods. These days it also requires significant computer resources for both algebraic and numerical calculations. Only a few groups of workers worldwide have the necessary expertise. My own role in this tennis game is purely that of a spectator, but I have been watching the ball bounce for some time, and I would like to give a brief account of the game from a fan's point of view, emphasizing the action on the theoretical side of the net.

The study of g is not just an exercise in accumulating decimal places for their own sake. The g factor represents an important test for fundamental theories of the forces of nature. So far, theory and experiment are in excellent agreement on the g factor of the electron. But for the muon—the heavier sibling of the electron—the situation is not so clear. Calculations and measurements of the muon g factor have not yet reached the precision of the electron results, but already there are hints of possible discrepancies. Those hints could be early signs of "new physics." Or they could be signs that we don't understand the old physics as well as we think we do.

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QED

The naive mental picture of an electron is a blob of mass and electric charge, spinning on its axis like a tiny planet. If we take this image seriously, the moving charge on the spinning particle's surface has to be regarded as an electric current, which ought to generate a magnetic field. The g factor (also known as the gyromagnetic ratio) is the constant that determines how much magnetic field arises from a given amount of charge, mass and spin. The formula is:

$$\mu = g\,\frac{e}{2m}\,s,$$

where μ is the magnetic moment, e the electric charge, m the mass and s the spin angular momentum (all expressed in appropriate units). Early experimental evidence suggested that the numerical value of g is approximately 2.

In the 1920s P. A. M. Dirac created a new and not-so-naive theory of electrons in which g was no longer just an arbitrary constant to be measured experimentally; instead, the value of g was specified directly by the theory. For an electron in total isolation, Dirac calculated that g is exactly 2. We now know that this result was slightly off the mark; g is greater than 2 by roughly one part in a thousand. And yet Dirac's mathematics was not wrong. The source of the error is that no electron is ever truly alone; even in a perfect vacuum, an electron is wrapped in a halo of particles and antiparticles, which are continually being emitted and absorbed, created and annihilated. Interactions with these "virtual" particles alter various properties of the electron, including the g factor.

Methods for accurately calculating g were devised in the 1940s as part of a thorough overhaul of the theory of electrons—a theory called quantum electrodynamics, or QED. That the calculation of g can be honed to such a razor edge of precision is something of a fluke. The mass, charge and magnetic moment of the electron are known only to much lower accuracy; so how can g, which is defined in terms of these quantities, be pinned down more closely? The answer is that g is a dimensionless ratio, calculated and measured in such a way that uncertainties in all those other factors cancel out.

Experimental measurements of g benefit from another fortunate circumstance. The experiments can be arranged to determine not g itself but the difference between g and 2; thus the measurements have come to be known as "g minus 2 experiments." Because $g-2$ is only about a thousandth of g, the measurement gains three decimal places of precision for free.

Bottled Electrons

One good way to measure g is to capture an electron and keep it in a bottle formed out of electric and magnetic fields. In confinement, the electron executes an elegant dance of twirls and pirouettes. The various modes of motion are quantized, meaning that only certain discrete energy states are possible. In some of these states the electron's intrinsic magnetic moment is oriented parallel to the external magnetic field, and in other states it is antiparallel. The energy difference between two such states is proportional to g. Thus a direct approach to determining g is simply to measure the energy of a "spin-flip" transition between parallel and antiparallel states.

The drawback of this straightforward experimental design is that you cannot know g with any greater accuracy than you know the strength of the external field. A clever trick sidesteps this problem: Measure the energies of *two* transitions, both of which depend on the magnetic field but only one of which involves a spin flip. For the non-flip transition, the constant of proportionality that sets the energy scale is exactly 2, whereas for the spin-flip transition the constant is g. Taking the ratio of the two energies eliminates dependence on the strength of the field.

Experiments with isolated electrons were pioneered by Hans Dehmelt of the University of Washington, who kept them penned up for weeks at a time—long enough that some of them were given names, like family pets. Although the technique may sound simple in its principles, getting results accurate to 11 significant figures is not a project for a high school science fair.

In the case of the muon, measuring g is even more challenging. The muon is an unstable particle, with a lifetime of a few microseconds, and so keeping a pet muon in a cage is not an option. The best muon $g-2$ measurements come from a 20-year-long experiment designated E821, carried out at the Brookhaven National Laboratory by workers from 11 institutions. Clouds of muons with their spins aligned circulate in a toroidal vacuum chamber immersed in a strong magnetic field. The apparatus is adjusted so that if g were exactly 2, the particles would complete each orbit with the same orientation they had at the outset. But because g differs from 2, the spin axis precesses slowly, drifting about 0.8 degree on each circuit of the ring. When a muon decays, it emits an electron preferentially in the direction of the spin axis. The spatial distribution of these electrons reveals the rate of precession and thus the value of $g-2$.

The latest value of the muon g factor reported by the E821 group works out to 2.0023318416 ± 0.0000000012. This number differs from the electron g factor in the fifth decimal place, and its precision is only at the parts-per-billion level rather than parts-per-trillion. Despite the lesser precision, however, the confrontation between theory and experiment turns out to be more dramatic in the case of the muon.

g-Whiz

Calculating g from theoretical principles might seem to be far easier than measuring it experimentally. After all, the theorist can leave behind all the messy imperfections of the physical world and operate in an abstract realm where vacuums and magnetic fields are always ideal, and no one ever spills coffee on the control panel. But theory has challenges of its own, and in the saga of the g factor, 20-year-long experiments are matched by 30-year-long calculations.

What needs to be calculated is the strength of a charged particle's interaction with a magnetic

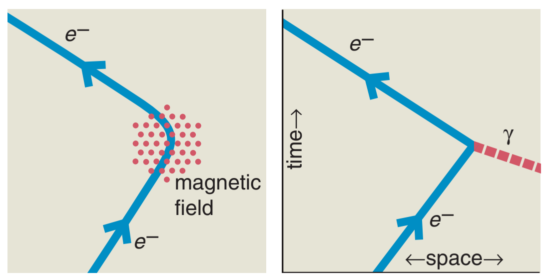

Figure 1. The g factor sets the strength of an electron's interaction with a magnetic field. In classical physics *(left)* magnetic lines of force (perpendicular to the page) induce a curvature in the electron's path. In quantum electrodynamics *(right)* the electron interacts with the field by emitting or absorbing a photon (γ). The event is represented in a Feynman diagram, where space extends along the horizontal axis and time moves up the vertical axis.

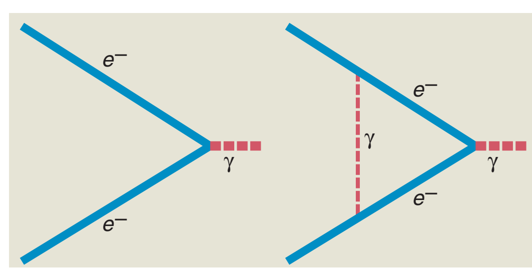

Figure 2. Feynman diagram is not just a picture of an event but also a computational device for keeping track of all the ways the event might happen. For electron-photon scattering, the tree-like diagram at left is not the only possibility. The electron can also emit and then reabsorb a "virtual" photon, as in the one-loop diagram at right. Here and in subsequent figures, Feynman diagrams are shown in a stylized form where only the topology—the pattern of connections between vertices—is meaningful.

field. The problem can be phrased in terms of something directly observable: Given a particle of known mass, charge and momentum, and a magnetic field of known intensity, how much will the particle's path be deflected when it passes through the field? Classical physics envisions magnetic lines of flux that induce a curvature in the particle's trajectory. Quantum electrodynamics takes a different approach. Instead of a field exerting its influence throughout a volume of space, QED posits a discrete, localized "scattering event," where an electron either emits or absorbs a photon (the quantum of the electromagnetic field); the recoil from this emission or absorption alters the electron's own motion.

A key tool for understanding such scattering events is the diagrammatic method introduced in the 1940s by Richard P. Feynman. A Feynman diagram plots position in space along one axis and time along another, so that a particle moving with constant velocity is represented by an oblique straight line. The Feynman diagram for a simple scattering event might have an electron moving diagonally until it collides with a photon coming from the opposite direction; at this "vertex" of the diagram the photon disappears and the electron reverses course.

There is more to a Feynman diagram, however, than just a spacetime depiction of particles colliding like billiard balls. As a matter of fact, in QED a particle cannot be assigned a unique, definite trajectory; all you can calculate is the probability that the particle will make its way from point *A* to point *B*. A Feynman diagram represents an entire family of possible trajectories, corresponding to collisions taking place at various positions and times. Each such trajectory has an associated "amplitude"; adding all the amplitudes and squaring the result yields the probability for the overall process.

The simplest scattering event—one electron bouncing off one photon—was the process considered by Dirac in his first computation of g in the 1920s. As noted above, Dirac got an exact result of $g = 2$. The reason this value needs correcting is that the simplest, one-photon scattering process is not the only way for an electron to get from point *A* to point *B*. The direct route may well be the most important path, but in QED you dare not ignore detours or distractions along the way.

One such distraction is for the electron to emit a photon and then reabsorb it, somewhat like a child throwing a ball in the air and running to catch it herself. The evanescent photon is called a virtual particle, because it can never be detected directly, but its effects on g are certainly real. Adding a virtual photon to the Feynman diagram is easy enough—it forms a loop, diverging from and then rejoining the electron path—but computing the photon's effect on g is more difficult. The problem is that the virtual photon can have unlimited energy. For an accurate computation, you have to add up the amplitudes associated with every possible energy—and without an upper limit, this sum comes out infinite. These implausibly infinite answers stymied the further development of QED for two decades.

The solution was a trick called renormalization, worked out by Feynman, Julian Schwinger, Sin-Itiro Tomonaga and Freeman Dyson. In 1947 Schwinger finally succeeded in calculating the contribution of a single virtual-photon loop to the g factor of the electron. The answer was given in terms of another fundamental constant of nature, known as α, which measures the electric charge of the electron and has a numerical value of about $1/137$. Schwinger showed that the one-loop contribution to the "anomalous magnetic moment" of the electron is $\alpha/2\pi$, or approximately 0.00116. The anomalous magnetic moment is defined as one-half of $g-2$, and so the corrected value of g comes out to about 2.00232.

g-Willikers

If an electron can get away with spontaneously tossing around a virtual photon, what's to stop it from juggling two or three of them? Nothing at all: A Feynman diagram decorated with a single photon loop can just as well be festooned with two loops. Furthermore, it turns out there are seven distinct two-loop diagrams (*see Figure 3*).

Drawing the seven two-loop Feynman diagrams is actually the easy part of understanding their effect; the hard part is calculating each diagram's contribution to the value of g. The mathematical expression associated with a diagram takes the form of an integral, summing up the amplitudes of an infinite family of particle paths. Some of the two-loop integrals are complicated, and early attempts to evaluate them went astray; the task was not completed until 1957. The result is again expressed in terms of α, but—reflecting the much lower probability of a two-loop event—the α term is now squared. And it is multiplied by

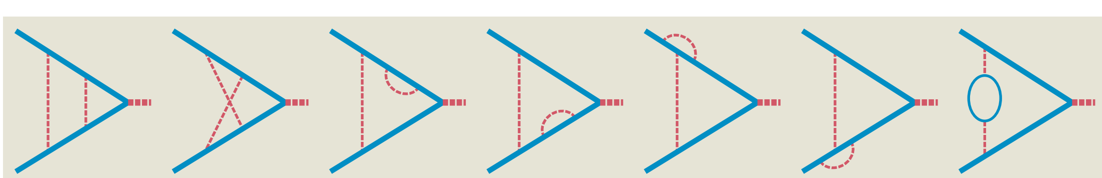

Figure 3. Two-loop Feynman diagrams offer seven more variations on the theme of an electron scattered by a photon. Electrons (and their antiparticles, positrons) are represented by solid blue lines; photons are dashed red lines. Real particles (those that can be observed directly in the laboratory) are shown as heavy lines, virtual particles as finer lines. Six of the diagrams have two virtual photons; in the last case, a virtual photon gives rise to a virtual electron-positron pair..

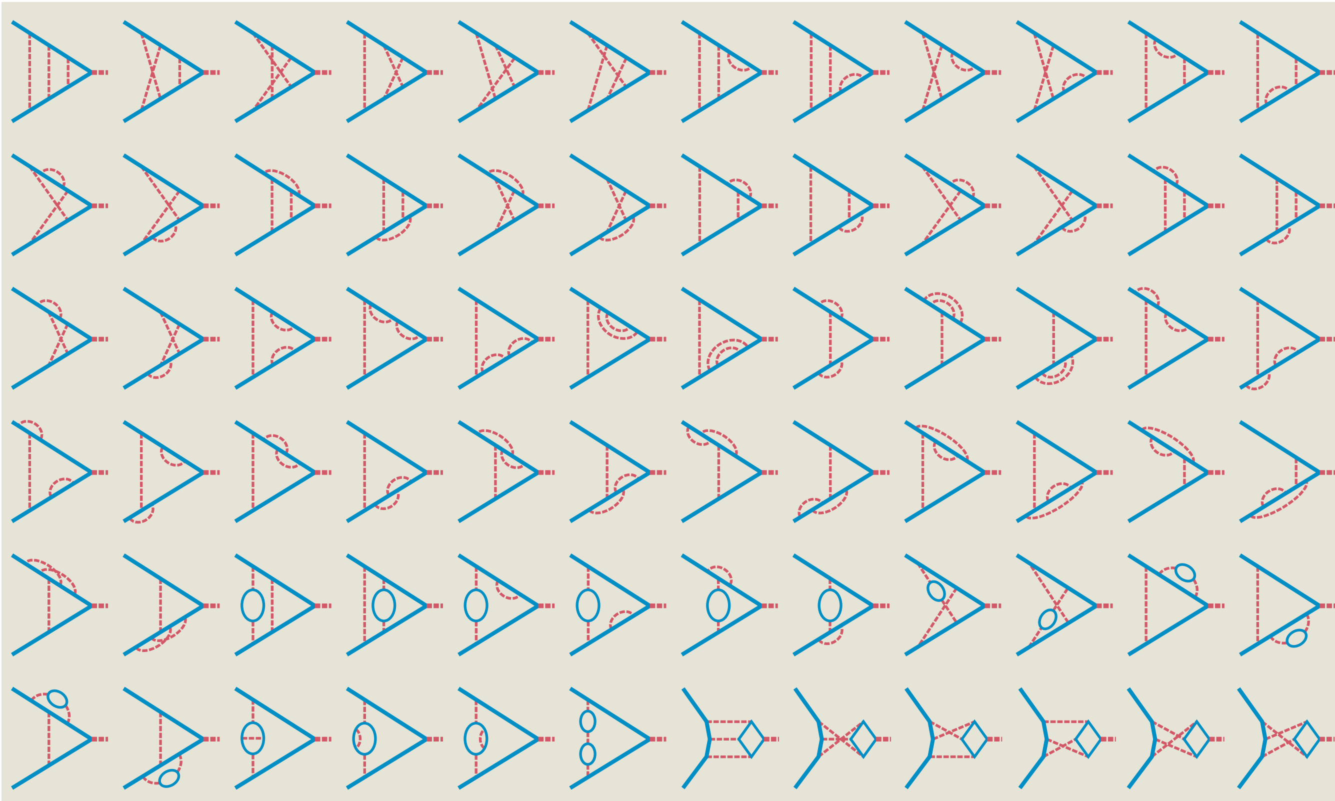

Figure 4. Seventy-two three-loop diagrams contribute to the calculation of the electron *g* factor. Fifty of these diagrams involve three virtual photons, and another 16 diagrams include an electron-positron bubble. The last six diagrams represent "light-by-light" scattering, in which the real electron and photon never come in direct contact; their interaction is mediated by a triplet of virtual photons and a virtual electron-positron pair. In the calculation of the muon *g* factor, a somewhat different kind of light-by-light scattering occasioned an error that took six years to track down.

a curious coefficient that combines various rational fractions, logarithms and the Riemann zeta function—this last item being familiar in number theory but an exotic interloper in physics.

What comes next is no surprise: If two loops are good, three must be better. However, for an electron-photon event with three loops there are 72 Feynman diagrams, representing integrals of daunting difficulty (*see Figure 4*). When work on evaluating those integrals got under way in the 1960s, it soon became clear that the methods of pencil-and-paper algebra had reached their limits. In this way Feynman-diagram calculations became a major impetus to the development of computer-algebra systems—programs that can manipulate and simplify symbolic expressions.

Despite such computational power tools, some of the three-loop diagrams resisted analytic solution for 30 years. To fill in the gaps, physicists tried numerical methods of evaluating the integrals—an even more computer-intensive task. A simple example of numerical integration is estimating the area of a geometric figure by randomly throwing darts at it and counting the hits and misses. The same basic idea can be applied to a Feynman integral, but the object being measured is now a complicated volume in a high-dimensional space; this makes the dart-throwing process painfully inefficient. Merely deciding whether or not a dart has hit the target becomes time-consuming. It was not until 1995 that a reliable, high-precision value of the three-loop contribution was published, by Toichiro Kinoshita of Cornell University. He evaluated all 72 diagrams numerically, comparing and combining his results with analytic values that were then known for 67 of the diagrams. A year later the last few diagrams were solved analytically by Stefano Laporta and Ettore Remiddi of the University of Bologna.

The three-loop correction is proportional to α^3, which makes its order of magnitude less than one part per million. Even so, to match the precision of the experimental measurement it's necessary to go on to the four-loop diagrams, of which there are 891. Attacking all those intricately tangled diagrams by analytic methods is hopeless for now. Numerical computations have been under way since the early 1980s. A thousandfold increase in the computer time invested in the task has brought a thirtyfold improvement in precision—but the best results still amount to only a few significant digits.

The Muon's Story

The electron and the muon are twins (or triplets, since there is a third sibling called the tau). The only apparent difference between them is mass, the muon being 200 times as heavy. But mass matters mightily in the calculation of *g*. Because certain effects are proportional to the square of the mass, they are enhanced 40,000 times in the muon. As a result, the muon *g* factor depends not just on electromagnetic interactions but also on manifestations of the weak and the strong nuclear forces. The virtual particles that appear in muon Feynman diagrams include the usual photons and electrons and also heavier objects such as the *W* and *Z* (quanta of the weak force) and the strongly interacting particles known collectively as hadrons.

A theoretical framework called the Standard Model extends the ideas of QED to the strong and weak forces. Unfortunately, however, the theory does not always allow high-precision calculations from first principles in the way QED does. The strong-force contributions have to be computed on a more empirical basis; in effect, even the theoretical value of the muon *g* factor is based in part on experimental findings.

The muon *g* factor has attracted much attention lately because the theoretical and experimental values seem to be diverging. The latest measurements from the E821 group differ from accepted theoretical values by roughly two standard deviations. Physicists have not been reticent about speculating on the possible meaning of this discrepancy, suggesting it could be our first glimpse of physics beyond the Standard Model. Perhaps the muon is not truly an elementary particle but has some kind of substructure? Another popular notion is supersymmetry, which predicts that all particles have shadowy "superpartners," with names such as selectrons, smuons and photinos.

One of these adventurous interpretations of the muon results could well turn out to be true. On the other hand, it seems prudent to keep in mind that the *g*-factor experiments and calculations are fearfully difficult, and it's always possible an error has crept in somewhere along the way. It would not be the first time. Feynman, in his book *QED: The Strange Theory of Light and Matter*, tells the story of an early computation of the two-loop electron *g* factor:

> It took two 'independent' groups of physicists two years to calculate this next term, and then another year to find out there was a mistake—experimenters had measured the value to be slightly different, and it looked for a while that the theory didn't agree with experiment for the first time, but no: it was a mistake in arithmetic. How could two groups make the same mistake? It turns out that near the end of the calculation the two groups compared notes and ironed out the differences between their calculations, so they were not really independent.

The story has been re-enacted more recently. In the mid-1990s two groups independently calculated a small, troublesome contribution to the muon *g* factor called hadronic light-by-light scattering. Kinoshita's group and a European collaboration of Johan Bijnens, Elisabetta Pallante and Joaquím Prades got compatible results. Then, six years later, Marc Knecht and Andreas Nyffeler recalculated the effect by another method and came up with an answer that was the same in magnitude but opposite in sign. The other groups rechecked their work, and both found they had made essentially the same mistake entering formulas into a computer-algebra program. The correction slightly diminished the disagreement between theory and experiment.

In mentioning such incidents, my aim is certainly not to embarrass the participants. They are working far out on the frontier of computational science, where no maps or signposts show the way. But for that very reason a certain amount of caution is in order when evaluating the results.

A definitive understanding of the muon *g* factor will have to await further refinements of both the experimental and the theoretical values. Incremental improvements can be expected soon, but major advances may be some time in coming. On the experimental side, the E821 project has been shut down by the Department of Energy, at least for the time being. As for theory, the next major stage will require serious attention to the five-loop Feynman diagrams. There are 12,672 of those. Don't hold your breath.

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Note: A more extensive bibliography is available at http://www.americanscientist.org/ AssetDetail/assetid/32262

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